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GENERAL SOLUTIONS OF PROBLEMS OF THE THEORY OF ELASTICITY AND BOUNDARY-VALUE PROBLEMS†

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The vector potentials of the displacements of the general solutions of static Boussinesq and Papkovich problems are presented in a form which leads to the splitting of the vector equations of the potentials in cylindrical and spherical coordinates into two scalar potentials. The solutions of the equations of the scalar potentials for finite bodies of canonical form contain orthogonal systems of functions on the coordinate surfaces in the region occupied by the body considered, including its boundary surfaces. One thereby creates the prerequisites for converting the boundary conditions into infinite systems of linear algebraic equations after expanding the stresses or displacements, specified on the boundary surfaces, in orthogonal functions of the equations of the potentials.

1. INTRODUCTION

We mean by the general solution of problems of the theory of elasticity in displacements the replacement of the Lamé equation

$$\frac{1-\nu}{(1+\nu)(1-2\nu)}\operatorname{grad}\operatorname{div}\mathbf{u}-\frac{1}{2(1+\nu)}\operatorname{rot}\operatorname{rot}\mathbf{u}=\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1.1}$$

(ν is Poisson's ratio, $\mathbf{u}$ is the dimensionless displacement vector, t is the dimensionless time, referred to l/c, l is the characteristic linear dimension and c is the velocity of sound in the elastic medium), by simpler equations for the vector and scalar potentials of the displacement vector $\mathbf{u}$: by the vector and scalar wave equations in non-stationary dynamic problems, by the vector and scalar Helmholtz equations in stationary dynamic problems, and by the vector and scalar Laplace equations and by biharmonic equations in static problems.

The general Boussinesq solution of static problems of the theory of elasticity can be reduced to the vector equation [1–3]

$$\nabla^2\nabla^2\mathbf{G}=0 \tag{1.2}$$

Equation (1.1) (when $\partial^2\mathbf{u}/\partial t^2=0$) is satisfied if

$$\mathbf{u}=\frac{1-2\nu}{2(1-\nu)}\operatorname{grad}\operatorname{div}\mathbf{G}-\operatorname{rot}\operatorname{rot}\mathbf{G} \tag{1.3}$$

The general Papkovich solution of static problems reduces to vector and scalar Laplace equations [1, 2, 4]

$$\nabla^2\mathbf{B}=0,\quad \nabla^2\Phi=0 \tag{1.4}$$

while the displacement vector can be written in the form

$$\mathbf{u}=\mathbf{B}-\frac{1}{4(1-\nu)}\operatorname{grad}(\mathbf{rB}+\Phi) \tag{1.5}$$

Here $\mathbf{r}$ is the radius vector of the point at which the displacement $\mathbf{u}$ is defined.

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Solution (1.2), (1.3) was obtained for the first time by Boussinesq [3]. Assuming the rectangular coordinates x_1, x_2, x_3 to be dimensionless, the equilibrium equations in displacements, corresponding to vector equation (1.1), when taking volume forces into account, can be written in the form

$$\frac{1}{1-2\nu}\frac{\partial\Delta}{\partial x_k}+\nabla^2 u_k+2(1+\nu)\frac{1}{E}X_k=0, \quad k=1,2,3 \tag{1.6}$$

Here Δ is the volume deformation, u_k and X_k are the components of the vectors of the dimensionless displacement and the volume force respectively, and E is the modulus of elasticity.

Boussinesq introduced three functions of the coordinates ψ_1, ψ_2, ψ_3 and represented the components of the displacements in the form

$$u_k=-\frac{\partial H}{\partial x_k}+\nabla^2\psi_k, \qquad k=1,2,3 \tag{1.7}$$

Substitution of (1.7) into the system of three differential equations (1.6) leads to replacement of it by the following three equations

$$\nabla^2\nabla^2\psi_k=-2(1+\nu)\frac{1}{E}X_k, \quad k=1,2,3 \tag{1.8}$$

Here

$$2(1-\nu)\nabla^2 H=\frac{\partial}{\partial x_1}\nabla^2\psi_1+\frac{\partial}{\partial x_2}\nabla^2\psi_2+\frac{\partial}{\partial x_3}\nabla^2\psi_3$$

whence it follows that

$$H=\frac{1}{2(1-\nu)}\left(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}+\frac{\partial\psi_3}{\partial x_3}\right)+\Phi \tag{1.9}$$

Here Φ is an arbitrary harmonic function. The advisability of retaining the function Φ in expression (1.9) is extremely doubtful, since the overall order of Eqs (1.8) is quite high (see Section 5).

When $X_k = 0$, Eqs (1.8) correspond to vector equation (1.2), while relations (1.7), taking expression (1.9) into account when $\Phi = 0$, correspond to the vector equality (1.3).

A similar solution of the homogeneous problem was published much earlier by Galerkin [5], pointing out, without proof, the generality of solution (1.2), (1.3).

Solution (1.4), (1.5) was obtained by Papkovich [4]. A paper with a similar solution was later published by Neuber [6]. As Papkovich wrote [2, 4], solution (1.4), (1.5) was obtained earlier by Grodskii, but was published later [7].

Introducing the notation $\nabla^2\psi_k = B_k$ ($k = 1, 2, 3$) and substituting it into relations (1.7) in the case of the homogeneous problem, when $\nabla^2 B_k = 0$ ($k = 1, 2, 3$), we obtain, by substituting expressions (1.7) into (1.6)

$$2(1-\nu)\nabla^2 H=\frac{\partial B_1}{\partial x_1}+\frac{\partial B_2}{\partial x_2}+\frac{\partial B_3}{\partial x_3}$$

whence it follows that

$$H=\frac{1}{4(1-\nu)}(x_1B_1+x_2B_2+x_3B_3+\Phi) \tag{1.10}$$

where Φ is an arbitrary harmonic function. The vector equation (1.5) corresponds to relations (1.7) if we take expression (1.10) into account. Hence, the general Papkovich solution reduces to four Laplace equations instead of three biharmonic equations in the general Boussinesq solution.

Papkovich [2] gave a thorough analysis of the general Boussinesq, general Papkovich and some other solutions, found a relation between them and proved their redundant generality. In particular, he did not consider it necessary to retain the harmonic function Φ in relation (1.5) to keep the solution general, but he retained it in order to make it easier to satisfy the boundary conditions. Unlike the general

Boussinesq solution the introduction of the harmonic function Φ into the general Papkovich solution, as can be seen from the following (Sections 2 and 3), is not only useful but also necessary.

Solutions of specific problems of the theory of elasticity for the simplest finite bodies were constructed [1, 8, 9, etc.] using particular solutions of the equations of the displacement potentials. Complete solutions (a complete se of particular solutions) of the equations of the stationary dynamic problem of the theory of elasticity for finite bodies of canonical form in rectangular, cylindrical and spherical coordinates were given in [10, 11], which enable problems of the frequencies and forms of natural oscillations to be solved. The approximation of complex bodies by canonical bodies solves the important technical problem of calculating the natural frequencies of oscillations of the elements of structures using the equations of the theory of elasticity [12].

General solutions of static problems of the theory of elasticity by methods employed previously in [10, 11], can be reduced to a form which enables boundary-value problems to be solved for finite bodies of canonical form. A canonical body, as previously [10, 11], is a finite body obtained by the intersection of no more than three pairs of surfaces, where the surfaces of each pair belong to one of three families of coordinate surfaces.

2. REPRESENTATION OF THE GENERAL SOLUTIONS IN CYLINDRICAL COORDINATES

A canonical body with dimensionless cylindrical coordinates ρ, φ, z is a body obtained by the intersection of cylindrical surfaces $\rho = \rho_1$ and $\rho = \rho_2$, the half-planes $\varphi = 0$ and $\varphi = \varphi_0$ and the planes $z = 0$ and $z = z_0$ $(\rho_1 \leqslant \rho \leqslant \rho_2, 0 \leqslant \varphi \leqslant \varphi_0, 0 \leqslant z \leqslant z_0)$. A circular cylinder $(\rho_1 \leqslant \rho \leqslant \rho_2, 0 \leqslant z \leqslant z_0)$ is also a canonical body.

By analogy with the vector potential of transverse waves in the stationary dynamic problem [13], the biharmonic vector of the general Boussinesq solution can be written in the form

$$\mathbf{G} = \psi \mathbf{e}_z + \operatorname{rot}(\psi_* \mathbf{e}_z) \tag{2.1}$$

Here $\mathbf{e}_z$ is the unit vector along the z axis, and ψ and ψ_* are scalar functions of the coordinates. (The possibility of such an approach to static problems in cylindrical coordinates for the general Boussinesq solution was indicated previously in [14].)

Substituting expression (2.1) into (1.2) we obtain

$$\nabla^2\nabla^2\psi = 0, \qquad \nabla^2\nabla^2\psi_* = 0 \tag{2.2}$$

We introduce the notation

$$\nabla^2\psi = \psi_1, \qquad -\frac{1}{2(1-\nu)}\frac{\partial\psi}{\partial z} = F, \qquad \nabla^2\psi_* = \psi_2 \tag{2.3}$$

The displacement vector $\mathbf{u}$, after substituting expression (2.1) into Eq. (1.3) and taking the notation (2.3) into account, is given by

$$\mathbf{u} = \operatorname{grad} F + \psi_1 \mathbf{e}_z + \operatorname{rot}(\psi_2 \mathbf{e}_z) \tag{2.4}$$

Hence, the vector potential $\mathbf{G}$ of the displacements is replaced by scalar biharmonic potentials ψ and harmonic potentials ψ_2.

Laplace's equation, like Helmholtz' equation in the stationary dynamic problem [10, 11], can be solved by the method of separation of variables. The parameters of the separation are chosen so that, of the three ordinary differential equations, two are the Sturm–Liouville equations and, with the appropriate boundary conditions, describe the Sturm–Liouville problem in the region occupied by the canonical body considered. An alternate choice of the separation parameters, giving three possible combinations of two variables, by means of which the Sturm–Liouville problem is solved, leads to the construction of solutions of Laplace's equations and of the biharmonic equation in the form

$$\begin{gathered}\psi_1 = \sum_m\sum_n R_{mn}^{(1)} u_m \upsilon_n + \sum_m\sum_k Z_{mk}^{(1)} u_m w_{km} + \sum_l\sum_n \Phi_{ln}^{(1)} \upsilon_n t_{ln} \\ \psi_2 = \sum_m\sum_n R_{mn}^{(2)} \frac{1}{\mu_m}\frac{du_m}{d\varphi}\frac{1}{\nu_n}\frac{d\upsilon_n}{dz} + \sum_k\sum_m Z_{km}^{(2)} \frac{1}{\mu_m}\frac{du_m}{d\varphi} w_{km} + \sum_l\sum_n \Phi_{ln}\frac{1}{\upsilon_n}\frac{d\upsilon_n}{dz} t_{ln}\end{gathered} \tag{2.5}$$

$$\psi=\sum_m\sum_n(R_{mn}+R_{mn}^{(0)})u_m\upsilon_n+\sum_k\sum_m(Z_{km}+Z_{km}^{(0)})u_m w_{km}+\sum_l\sum_n(\Phi_{ln}+\Phi_{ln}^{(0)})\upsilon_n t_{ln}$$

Here $u_m(\varphi)$, $\upsilon_n(z)$, $w_{km}(\rho)$, $t_{ln}(\rho)$ are the eigenfunctions of the Sturm–Liouville equations

$$u_m=\cos\mu_m\varphi,\quad \mu_m=\frac{(m-1)\pi}{\varphi_0},\quad m=1,2,\ldots$$

$$\upsilon_n=\cos\nu_n z,\quad \upsilon_n=\frac{(n-1)\pi}{z_0},\quad n=1,2,\ldots \tag{2.6}$$

$$w_{km}=\frac{dY_\mu}{d\rho}(\rho_2h_{km})J_\mu(\rho h_{km})-\frac{dJ_\mu}{d\rho}(\rho_2h_{km})Y_\mu(\rho h_{km})$$

$$t_{1ln}=\operatorname{Re}I_{i\tau}(\rho\nu_n),\qquad t_{2ln}=\operatorname{Im}I_{i\tau}(\rho\nu_n)$$

$$t_{ln}=\frac{dt_{2ln}}{d\rho}(\rho_2\nu_n)t_{1ln}(\rho\nu_n)-\frac{dt_{1ln}}{d\rho}(\rho_2\nu_n)t_{2ln}(\rho\nu_n),\quad n>1 \tag{2.7}$$

$$t_{l1}=\cos\left(\tau_{l1}\ln\frac{\rho_2}{\rho}\right),\qquad \tau_{l1}=\frac{(l-1)\pi}{\ln(\rho_2/\rho_1)},\quad l=1,2,\ldots$$

Y_μ and J_μ are Bessel functions of the first and second kind respectively of order μ_m (the subscript m is omitted for brevity), $I_{i\tau}$ is the modified Bessel function of imaginary order $i\tau_{in}$, i is the square root of −1, and τ_{in} is the order modulus (the subscripts l and n are omitted for brevity).

The eigenvalues of the Sturm–Liouville problem h_{km} and τ_{ln} ($n>1$) are found respectively from the equations

$$\frac{dw_{km}}{d\rho}(\rho_1)=0,\quad k=1,2,\ldots\quad (h_{11}=0,\ w_{11}=1);\qquad \frac{dt_{ln}}{d\rho}(\rho_1)=0,\quad n>1 \tag{2.8}$$

The functions u_m and υ_n form an orthogonal system of functions on the cylindrical surfaces $\rho=\text{const}$, including the cylindrical boundaries $\rho=\rho_1$ and $\rho=\rho_2$ ($0\leqslant z\leqslant z_0$, $0\leqslant\varphi\leqslant\varphi_0$).

The functions w_{km} satisfy the orthogonality condition

$$\int_{\rho_1}^{\rho_2}w_{km}w_{lm}\rho d\rho=0,\qquad k\neq l$$

and, together with the function u_m, form an orthogonal system of functions in the $z=\text{const}$ planes, including the plane boundaries $z=0$ and $z=z_0$ ($\rho_1\leqslant\rho\leqslant\rho_2$, $0\leqslant\varphi\leqslant\varphi_0$).

The functions t_{ln} satisfy the orthogonality condition

$$\int_{\rho_1}^{\rho_2}t_{ln}t_{kn}\frac{1}{\rho}d\rho=0,\qquad l\neq k$$

and together with the functions υ_n form an orthogonal system of functions in the half-planes $\varphi=\text{const}$, including the boundaries $\varphi=0$ and $\varphi=\varphi_0$ ($\rho_1\leqslant\rho\leqslant\rho_2$, $0\leqslant z\leqslant z_0$).

The procedure for separating the variables in Laplace's equation also gives a third function, corresponding to each of the three combinations of two orthogonal functions

$$R_{mn}^{(j)}=A_{mn}^{(j)}\frac{I_\mu(\rho\nu_n)}{I_\mu(\rho_2\nu_n)}+B_{mn}^{(j)}\frac{K_\mu(\rho\nu_n)}{K_\mu(\rho_1\nu_n)},\quad n>1$$

$$R_{m1}^{(j)}=A_{m1}^{(j)}\rho^{\mu_m}+B_{m1}^{(j)}\rho^{-\mu_m},\quad m>1;\qquad R_{11}^{(j)}=A_{11}^{(j)}\ln\rho+B_{11}^{(j)}$$

$$Z_{km}^{(j)}=C_{km}^{(j)}\exp[-h_{km}(z_0-z)]+D_{km}^{(j)}\exp(-h_{km}z),\qquad Z_{11}^{(j)}=C_{11}^{(j)}z+D_{11}^{(j)}$$

$$\Phi_{ln}^{(j)}=E_{ln}^{(j)}\exp[-\tau_{ln}(\varphi_0-\varphi)]+G_{ln}^{(j)}\exp(-\tau_{ln}\varphi),\qquad \Phi_{11}^{(j)}=E_{11}^{(j)}\varphi+G_{11}^{(j)}$$

$$j=0,1,2$$

where I_μ and K_μ are modified Bessel functions of the first and second kind of order μ_m (the subscript m is omitted).

In the solution of the biharmonic equation, the third function

$$R_{mn} = -\frac{1}{2\nu_n^2}\rho\frac{dR_{mn}^{(1)}}{d\rho}, \qquad R_{m1} = \frac{1}{4(1-\mu_m^2)}\left(-\rho^2 R_{m1}^{(1)} + \rho^3\frac{dR_{m1}^{(1)}}{d\rho}\right)$$

$$Z_{km} = -\frac{1}{2h_{km}^2}z\frac{dZ_{km}^{(1)}}{dz}, \qquad Z_{11} = -\frac{1}{2}z^2 Z_{11}^{(1)} + \frac{1}{3}z^3\frac{dZ_{11}^{(1)}}{dz}$$

$$\Phi_{ln} = -\frac{1}{2\tau_{ln}^2}\varphi\frac{d\Phi_{ln}^{(1)}}{d\varphi}, \qquad \Phi_{11} = -\frac{1}{2}\varphi^2\Phi_{11}^{(1)} + \frac{1}{3}\varphi^3\frac{d\Phi_{11}^{(1)}}{d\varphi}$$

corresponds to these combinations of orthogonal functions, where $A_{mn}^{(j)}$, $B_{mn}^{(j)}$, $C_{mk}^{(j)}$, $D_{mk}^{(j)}$, $E_{ln}^{(j)}$, $G_{ln}^{(j)}$ $(j = 0, 1, 2)$ are sequences of arbitrary constants.

Introducing into (2.5) the conditions for the displacement potentials to be periodic with respect to the coordinate φ, we obtain the displacement potentials for a circular cylinder

$$\psi_1 = \sum_m\left(\sum_n R_{mn}^{(1)}\nu_n + \sum_k Z_{km}^{(1)}w_{km}\right)\cos(m-1)\varphi$$

$$\psi_2 = -\sum_m\left(\sum_n R_{mn}^{(2)}\frac{1}{\nu_n}\frac{d\nu_n}{dz} + \sum_k Z_{km}^{(1)}w_{km}\right)\sin(m-1)\varphi \tag{2.9}$$

$$\psi = \sum_m\left[\sum_n (R_{mn} + R_{mn}^{(0)})\nu_n + \sum_k (Z_{km} + Z_{km}^{(0)})w_{km}\right]\cos(m-1)\varphi$$

In the case of the axisymmetric problem ($m = 1$), solution (2.9) is identical with Abramyan's solution [15].

The representation of the displacement potentials in the form (2.5) and (2.9) predetermines the change of the boundary conditions into infinite systems of linear algebraic equations in the sequence of arbitrary constants. The free terms of the equations are the coefficients of the expansion of the stresses or displacements, specified on the boundary surfaces, in corresponding orthogonal systems of functions. The number of boundary conditions is equal to the number of sequences of arbitrary constants. When the system is truncated, i.e. on retaining the same number of terms with respect to each summation index, the number of unknowns corresponds to the number of equations.

The regularity of the infinite systems obtained is proved for the special case of axisymmetric deformation of a circular cylinder in [15].

The general Papkovich solution can be represented in a similar way. To do this the harmonic vector **B** is written in the same way as the biharmonic vector **G** (2.1)

$$\mathbf{B} = \psi_1\mathbf{e}_z + \operatorname{rot}(\psi_2\mathbf{e}_z) \tag{2.10}$$

after which the first vector equation of (1.4) changes into two scalar Laplace equations

$$\nabla^2\psi_1 = 0, \quad \nabla^2\psi_2 = 0 \tag{2.11}$$

Substituting expression (2.10) into (1.5) we obtain the displacement vector in the form (2.4) if we introduce the notation

$$F = -\frac{1}{4(1-\nu)}\left(\frac{\partial\psi_2}{\partial\varphi} + z\psi_1 + \Phi\right) \tag{2.12}$$

Although the functions F have a different form in the general Boussinesq and Papkovich solutions, the Laplacian of these functions is the same in both general solutions, due to the fact that the volume deformation

$$\Delta = \operatorname{div}\mathbf{u} = \nabla^2 F + \frac{\partial\psi_1}{\partial z} = -(1-2\nu)\nabla^2 F = \frac{1-2\nu}{2(1-\nu)}\frac{\partial\psi_1}{\partial z}$$

is identical. The elementary rotation vectors

$$\mathbf{w} = \operatorname{rot}\mathbf{u} = \operatorname{rot}(\psi_1\mathbf{e}_2) + \operatorname{grad}\frac{\partial\psi_2}{\partial z}$$

are also identical in both general solutions.

The function ψ_1 in relations (2.10) and (2.12) is given by the first relation of (2.5), while the function ψ_2 is best written in the form

$$\psi_2 = \sum_m \sum_n R_{mn}^{(2)} \frac{1}{\mu_m}\frac{du_m}{d\varphi} v_n + \sum_k\sum_m Z_{km}^{(2)} \frac{1}{\mu_m}\frac{du_m}{d\varphi} w_{km} + \sum_l\sum_n \Phi_{ln}^{(2)} v_n t_{ln}$$

The function Φ can be obtained from φ_1 by replacing $R_{mn}^{(1)}$, $Z_{km}^{(1)}$, $\Phi_{ln}^{(1)}$ by $R_{mn}^{(0)}$, $Z_{km}^{(0)}$, $\Phi_{ln}^{(0)}$ respectively.

3. REPRESENTATION OF THE GENERAL SOLUTIONS IN SPHERICAL COORDINATES

In dimensionless spherical coordinates ρ, θ, φ ($\rho, x, \varphi, x = \cos\theta$) the canonical body occupies a region formed by the intersection of the spherical surfaces $\rho = \rho_1$ and $\rho = \rho_2$ with the canonical surfaces $\theta = \theta_1$ and $\theta = \theta_2$ ($x = x_1$ and $x = x_2$) and the half-plane $\varphi = 0$ and $\varphi = \varphi_0$ ($\rho_1 \leqslant \rho \leqslant \rho_2$, $\theta_1 \leqslant \theta \leqslant \theta_2$, $0 \leqslant \varphi \leqslant \varphi_0$).

The solid of revolution obtained by the intersection of the canonical and spherical surfaces ($\rho_1 \leqslant \rho \leqslant \rho_2$, $\theta_1 \leqslant \theta \leqslant \theta_2$), and a hollow or solid sphere is also a canonical body.

The splitting of the vector equation (1.2) into two scalar equations is due to the representation of the biharmonic vector $\mathbf{G}$ in the form

$$\mathbf{G} = \rho\psi\mathbf{e}_\rho + \operatorname{rot}(\rho\psi_*\mathbf{e}_\rho) + \operatorname{grad}\psi_0 \tag{3.1}$$

Here $\mathbf{e}_\rho$ is the unit vector coinciding with the direction of variation of the coordinate ρ, and ψ, ψ_* and ψ_0 are scalar functions of the coordinates.

Unlike cylindrical coordinates, in spherical coordinates the analogy between (3.1) and the potential of the transverse waves in the stationary dynamic problem [13] is incomplete: in (3.1) an additional function ψ_0 has been introduced, which is a particular solution of the equation

$$\nabla^2\psi_0 = -4\psi \tag{3.2}$$

Substituting (3.1) into (1.2) we obtain

$$\nabla^2\nabla^2\psi = 0, \quad \nabla^2\nabla^2\psi_* = 0 \tag{3.3}$$

By introducing the notation

$$\nabla^2\psi = \psi_1, \quad -\frac{1-2\nu}{1-\nu}\psi - \frac{1}{2(1-\nu)}\frac{\partial}{\partial\rho}(\rho\psi) = F, \quad \nabla^2\psi_* = \psi_2 \tag{3.4}$$

and substituting (3.1) into (1.3), we can write the displacement vector in the form

$$\mathbf{u} = \operatorname{grad} F + \rho\psi_1\mathbf{e}_\rho + \operatorname{rot}(\rho\psi_2\mathbf{e}_\rho) \tag{3.5}$$

Hence, the general Boussinesq solution in spherical coordinates, as also in cylindrical coordinates, reduces to a biharmonic equation (the first equation of (3.3)) and the Laplace equation

$$\nabla^2\psi_2 = 0 \tag{3.6}$$

The solution of these equations in spherical coordinates is constructed using the same scheme as in cylindrical coordinates, and is written in the form

$$\psi_1 = \sum_m\sum_n R_{mn}^{(1)} u_m v_{mn} + \sum_k\sum_m X_{km}^{(1)} u_m w_k + \sum_k\sum_l \Phi_{kl}^{(1)} w_k t_{kl}$$

$$\psi_2 = \sum_m \sum_n R_{mn}^{(2)} \frac{1}{\mu_m} \frac{du_m}{d\varphi} v_{mn} + \sum_k \sum_m X_{km}^{(2)} \frac{1}{\mu_m} \frac{du_m}{d\varphi} w_k + \sum_k \sum_l \Phi_{kl}^{(2)} w_k t_{kl}$$

$$\psi = \sum_m \sum_n (R_{mn} + R_{mn}^{(0)}) u_m v_{mn} + \sum_k \sum_m (X_{km} + X_{km}^{(0)}) u_m w_k + \sum_k \sum_l (\Phi_{kl} + \Phi_{kl}^{(0)}) w_k t_{kl} \tag{3.7}$$

Here $u_m(\varphi)$, $v_{mn}(x)$, $w_k(\rho)$ and $t_{kl}(x)$ are orthogonal system of functions in the region occupied by the canonical body

$$u_m = \cos\mu_m\varphi,\quad \mu_m = \frac{(m-1)\pi}{\varphi_0},\quad m = 1,2,\ldots$$

$$v_{mn} = \frac{dQ_\nu^\mu}{dx}(x_2)P_\nu^\mu(x) + \frac{dP_\nu^\mu}{dx}(x_2)Q_\nu^\mu(x) \tag{3.8}$$

and $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are the associated Legendre functions of the first and second kind of order μ_m and degree ν_{nm}, respectively (the subscripts m and n are omitted for brevity). The degree ν_{nm} is the nth eigenvalue of the Sturm–Liouville problem and is given by the equation

$$\frac{dv_{mn}}{dx}(x_1) = 0,\quad \nu_{11} = 0,\quad v_{11} = 1$$

The functions $v_{nm}(x)$ satisfy the orthogonality condition

$$\int_{x_1}^{x_2} v_{mn} v_{ml} dx = 0,\quad n \neq l$$

and, together with the functions u_m, form an orthogonal system of functions on the spherical surfaces ρ = const ($x_1 \leq x \leq x_2$, $0 \leq \varphi \leq \varphi_0$), including the spherical boundaries $\rho = \rho_1$ and $\rho = \rho_2$.

The function $w_k(\rho)$ is given by the relation

$$w_k = \rho_2^{-1/2}\rho^{-1/2}\left[-\frac{1}{2}\sin\left(\tau_k \ln\frac{\rho_2}{\rho}\right) + \tau_k \cos\left(\tau_k \ln\frac{\rho_2}{\rho}\right)\right] \tag{3.9}$$

The eigenvalues τ_k are given by

$$\tau_k = \frac{(k-1)\pi}{\ln(\rho_2/\rho_1)},\quad k = 1,2,\ldots$$

The function w_k satisfies the orthogonality condition

$$\int_{\rho_1}^{\rho_2} w_k w_l d\rho = 0,\quad k \neq l$$

and, together with the functions u_m, form an orthogonal system on the canonical surfaces x = const ($\rho_1 \leq \rho \leq \rho_2$, $0 \leq \varphi \leq \varphi_0$), including the conical boundaries $x = x_1$ and $x = x_2$.

The function $t_{kl}(x)$ can be written in the form

$$t_{kl} = \frac{dt_{2kl}}{dx}(x_2)t_{1kl}(x) - \frac{dt_{1kl}}{dx}(x_2)t_{2kl}(x) \tag{3.10}$$

Here

$$t_{1kl} = \operatorname{Re} P_{-1/2+i\tau}^{i\zeta}(x),\quad t_{2kl} = \operatorname{Im} P_{-1/2+i\tau}^{i\zeta}(x)$$

where $P_{-1/2+i\tau}^{i\zeta}$ is the associated Legendre function of the first kind of imaginary order $i\zeta_{kl}$ and complex degree $-1/2 + i\tau_k$ (i is the square root of -1, and the subscripts l and k are omitted for brevity).

The modulus of order ζ_{kl} is the Sturm–Liouville eigenvalue and is given by the equation

$$\frac{dt_{kl}}{dx}(x_1)=0, \quad l=1,2,\ldots$$

The functions t_{kl} satisfy the orthogonality condition

$$\int_{x_1}^{x_2} t_{kl}t_{kn}\frac{1}{1-x^2}dx=0, \quad l\neq n$$

and together with the functions w_k form an orthogonal system on the half-planes φ = const ($\rho_1 \leqslant \rho \leqslant \rho_2, x_1 \leqslant x \leqslant x_2$), including on the plane boundaries $\varphi = 0$ and $\varphi = \varphi_1$.

When separating the variables in Laplace's equation, the third function

$$R_{mn}^{(j)} = A_{mn}^{(j)}\rho^{\nu_{mn}} + B_{mn}^{(j)}\rho^{-(\nu_{mn}+1)}$$
$$X_{km}^{(j)} = C_{km}^{(j)}X_{1km} + D_{km}^{(j)}X_{2km}$$
$$\Phi_{kl}^{(j)} = E_{kl}^{(j)}\exp[-\varsigma_{kl}(\varphi_0-\varphi)]+G_{kl}^{(j)}\exp(-\varsigma_{kl}\varphi)$$

corresponds to each pair of orthogonal functions. Here X_{1km} and X_{2km} are the particular solutions of the differential equation of the cone function [16]. The relation between the functions X_{1km} and X_{2km} and the associated Legendre functions is given by the expressions

$$P^{\mu}_{-\frac{1}{2}+i\tau} = X_{1km} - X_{2km}$$
$$Q^{\mu}_{-\frac{1}{2}+i\tau} = \frac{\pi}{4}\cos\pi\mu_m\left(\frac{1}{a_c}X_{2km}+\frac{1}{a_t}X_{1km}\right)+i\frac{\pi}{4}\left(\frac{1}{a_c}X_{2km}-\frac{1}{a_t}X_{1km}\right)$$
$$a_c = \mathrm{ch}^2\frac{\pi\tau_k}{2}-\cos^2\left[\frac{\pi}{2}\left(\frac{1}{2}-\mu_m\right)\right], \quad a_t = \mathrm{ch}^2\frac{\pi\tau_k}{2}-\sin^2\left[\frac{\pi}{2}\left(\frac{1}{2}-\mu_m\right)\right]$$

obtained from the values of the associated Legendre functions on the "branch cut", i.e. for the real argument $-1 < x < 1$ [16].

The functions X_{1km} and X_{2km} have the form

$$X_{rkm} = \frac{2^{\mu_m}\pi}{\gamma_{1km}(0)\gamma_{2km}(0)(1-x^2)^{\mu_m/2}}\sum_{p=0}^{\infty}\frac{\gamma_{rkm}(p)}{\Gamma(r-\frac{1}{2}+p)p!}x^{2p+r-1}, \quad r=1,2 \tag{3.11}$$
$$\gamma_{rkm}(p)=\left[\mathrm{Re}\,\Gamma\left(\frac{2r-1}{4}-\frac{\mu_m}{2}+p+\frac{1}{2}i\tau_k\right)\right]^2+\left[\mathrm{Im}\,\Gamma\left(\frac{2r-1}{4}-\frac{\mu_m}{2}+p+\frac{1}{2}i\tau_k\right)\right]^2$$

where Γ is the gamma function.

In the solution of the biharmonic equation the following third function corresponds to each pair of the same orthogonal functions

$$R_{mn} = -\frac{1}{2(2\nu_{mn}+3)(-2\nu_{mn}+1)}\left(\rho^2R_{mn}^{(1)}-2\rho^3\frac{dR_{mn}^{(1)}}{d\rho}\right)$$
$$X_{km} = -\rho^2\frac{\gamma_{1km}(0)\gamma_{2km}(0)}{\pi 2^{\mu_m+1}}[C_{km}^{(1)}(-JX_{1km}+J_1X_{2km})+D_{mk}^{(1)}(-J_2X_{1km}+JX_{2km})]$$
$$\Phi_{kl} = -\rho^2(1-x^2)\frac{1}{2\zeta_{kl}^2}\left(-\frac{1}{2}\Phi_{kl}^{(1)}+\varphi\frac{d\Phi_{kl}^{(1)}}{d\varphi}\right)$$
$$J=\int X_{1km}X_{2km}dx, \quad J_r=\int X_{rkm}^2dx, \quad r=1,2$$

where $A_{mn}^{(j)}, B_{mn}^{(j)}, C_{km}^{(j)}, D_{km}^{(j)}, E_{kl}^{(j)}, G_{kl}^{(j)}$ $(j = 0, 1, 2)$ are sequences of arbitrary constants.

For a solid of revolution, bounded by spherical and conical surfaces ($\rho_1 \leqslant \rho \leqslant \rho_2, x_1 \leqslant x \leqslant x_2$), the conditions of periodicity with respect to the coordinate φ are satisfied if μ_m is replaced by $m-1$

$$\psi_1=\sum_m\left(\sum_n R_{mn}^{(1)}\nu_{mn}+\sum_k Z_{km}^{(1)}w_{km}\right)\cos(m-1)\varphi$$
$$\psi_2=\sum_m\left(\sum_n R_{mn}^{(1)}\nu_{mn}+\sum_k Z_{km}^{(2)}w_{km}\right)\sin(m-1)\varphi \qquad (3.12)$$
$$\psi=\sum_m\left[\sum_n (R_{mn}+R_{mn}^{(0)})\nu_{mn}+\sum_k (X_{km}+X_{km}^{(0)})w_{km}\right]\cos(m-1)\varphi$$

In the general Papkovich solution the harmonic vector **B** is written by analogy with the biharmonic vector **G** (3.1) in the form

$$\mathbf{B}=\rho\psi_1\mathbf{e}_\rho+\operatorname{rot}(\rho\psi_2\mathbf{e}_\rho)+\operatorname{grad}\psi_0, \qquad (3.13)$$

where ψ_0 is the particular solution of the equation

$$\nabla^2\psi_0=-2\psi_1$$

When expression (2.13) is substituted into the first equation of (1.4), the vector Laplace equation splits into two scalar equations

$$\nabla^2\psi_1=0,\quad \nabla^2\psi_2=0 \qquad (3.14)$$

and the displacement vector **u** (1.5) takes the form (3.5) if we introduce the notation

$$F=\psi_0-\frac{1}{4(1-\nu)}\left(\rho^2\psi_1+\rho\frac{\partial\psi_0}{\partial\rho}+\Phi\right)$$

Just as in cylindrical coordinates, the Laplacian of the functions F

$$\nabla^2F=-2\psi_1-\frac{1}{2(1-\nu)}\frac{\partial}{\partial\rho}(\rho\psi_1)$$

is common in the general Boussinesq and Papkovich solutions, as a result of which the volume deformation Δ is identical in both general solutions

$$\Delta=\operatorname{div}\mathbf{u}=\frac{1-2\nu}{2(1-\nu)}\frac{\partial}{\partial\rho}(\rho\psi_1)$$

The vector **w** of elementary rotation is also expressed by the same relation in the general Boussinesq and Papkovich solutions

$$\mathbf{w}=\operatorname{rot}\mathbf{u}=\operatorname{rot}(\rho\psi_1\mathbf{e}_\rho)+\operatorname{grad}\frac{\partial}{\partial\rho}(\rho\psi_2)$$

4. THE BOUNDARY CONDITIONS OF THE STURM–LIOUVILLE PROBLEM

The eigenfunctions $y_m(x)$ ($m=1,2,\ldots$) of the Sturm–Liouville problem in the section $a\leqslant x\leqslant b$ satisfy the boundary conditions [17]

$$\alpha_1y_m(a)+\alpha_2y_m'(a)=0,\quad \beta_1y_m(b)+\beta_2y_m'(b)=0 \qquad (4.1)$$

where α_1, α_2, β_1, β_2 are constants, subject to the conditions

$$|\alpha_1|+|\alpha_2|>0,\quad |\beta_1|+|\beta_2|>0 \qquad (4.2)$$

Solutions (2.6), (2.7) and (3.8)–(3.10) of the Sturm–Liouville problems satisfy boundary conditions (4.1) when $\alpha_1=\beta_1=0$, $\alpha_2=\beta_2=1$. However, other combinations of the constants α_1, β_1, α_2, β_2 are possible.

It is appropriate to make the eigenfunctions of the Sturm–Liouville equation and the equation for the eigenvalues for arbitrary α_1, β_1, α_2, β_2 subject to conditions (4.2).

If $y_{1m}(x)$ and $y_{2m}(x)$ are two particular solutions of the Sturm–Liouville equation, corresponding to the eigenvalue λm, the eigenfunction $y_m(x)$ can be written in the form

$$y_m(x)=[\beta_1 y_{2m}(b)+\beta_2 y'_{2m}(b)]y_{1m}(x)-[\beta_1 y_{1m}(b)+\beta_2 y'_{1m}(b)]y_{2m}(x)$$

or

$$y_m(x)=[\alpha_1 y_{2m}(a)+\alpha_2 y'_{2m}(a)]y_{1m}(x)-[\alpha_1 y_{1m}(a)+\alpha_2 y'_{1m}(a)]y_{2m}(x)$$

The equation of the eigenvalues takes the form

$$\alpha_1\beta_1[y_{1m}(a)y_{2m}(b)-y_{2m}(a)y_{1m}(b)]+\alpha_1\beta_2[y_{1m}(a)y'_{2m}(b)-y_{2m}(a)y'_{1m}(b)]+$$
$$+\alpha_2\beta_1[y'_{1m}(a)y_{2m}(b)-y'_{2m}(a)y_{1m}(b)]+\alpha_2\beta_2[y'_{1m}(a)y'_{2m}(b)-y'_{2m}(a)y'_{1m}(b)]=0$$

The choice of the combinations of constants α_1, β_1, α_2, β_2 affects not only the eigenvalue spectrum and, consequently, the convergence of the expansions in eigenfunctions $y_m(x)$, but also the order of the truncated system of linear algebraic equations for a constant number of retained terms. More frequently the combinations $\alpha_1 = \beta_1 = 0$, $\alpha_2 = \beta_2 = 1$ and $\alpha_1 = \beta_1 = 1$, $\alpha_2 = \beta_2 = 0$ are used. The latter combination is a necessary condition for changing to a rational form of boundary conditions in the $z = 0$ and $z = z_0$ planes in cylindrical coordinates and on the spherical boundaries $\rho = \rho_1$ and $\rho = \rho_2$ in spherical coordinates [18].

5. CONCLUDING REMARKS

The representation of the vector biharmonic potential of the displacements in cylindrical and spherical coordinates by relations (2.1) and (3.1) and the vector harmonic potential by relations (2.10) and (3.13) reduces the general Boussinesq solution to two scalar biharmonic equations (2.2) and (3.3), and reduces the general Papkovich solution to scalar Laplace equations (1.5), (2.11) and (3.14). Here the redundant generality of the general Boussinesq solution is partially eliminated, while that to the general Papkovich solution is completely eliminated. The complete elimination of the redundant generality of the general Boussinesq solution is due to the dependences (2.4) and (3.5) of the displacement vectors on the first biharmonic function (2.2) and (3.3) and on the Laplacían of the second biharmonic function. Hence, the general Boussinesq solution is reduced to a scalar biharmonic equation and the Laplace equation.

The criterion of sufficient generality can be assumed to be the overall order of the differential equations of the scalar displacement potentials. If it is equal to 6, the number of sequences of arbitrary constants in (2.5) and in (3.7) is identical with the number of boundary conditions, namely, 18. In this case, when the infinite system is truncated, the number of unknowns is equal to the number of equations..

It should be noted that similar conversions of the general Boussinesq and Papkovich solutions can be carried out not only in a rectangular system of coordinates but also in elliptical cylindrical, parabolic cylindrical and conical coordinates [13].

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